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LETTER TO THE EDITOR

Non-zero quantum contribution to the soliton mass in the SUSY sine-Gordon model

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Abstract. Using an explicitly finite method which only needs the discrete levels of Schrödinger equations, the non-zero quantum contribution to the soliton mass in the SUSY sine-Gordon model is found.

The evaluation of the first quantum correction to the kink or soliton mass in bidimensional supersymmetric models has been an interesting subject over the last decade, starting from the work of Witten and Olive [1]. In particular, Schonfeld [2] and Kaul and Rajaraman [3] calculated the non-vanishing kink mass correction, paying especial attention to the problems associated with the divergent character of a typical 'one-loop' contribution. In any case both bosonic and fermionic quantum corrections to the mass of extended topological objects are calculated taking the physical system enclosed in a compact space of length L: the following task is the normal-mode-frequency sum over the inhomogeneous vacuum with a subtraction procedure for an identical contribution over the homogeneous one. Afterwards the renormalisation technique for a 'one-loop' order term is applied and finally the limit $L\to\infty$ allows us to retrieve the initial open situation. The details of a hard process such as the one mentioned can be found in [4].

However, in a recent article [5], we presented a unified calculation method for both bosonic and fermionic quantum corrections to the kink mass energies interpreted as a peculiar version of the Casimir effect. Moreover our procedure is an explicitly finite technique which only works with the discrete levels of Schrödinger equations, thus bypassing subtleties connected with boundary conditions or renormalisation counterterms. Going to the associated supersymmetric models this explicitly finite method should be a valuable tool in order to evaluate the non-vanishing mass correction within the SUSY context. We can even expound some observations about the relation between supersymmetry and the topological non-trivial sectors [6]. The point is that the Lagrangian shifted around the kink or soliton is not invariant under the supersymmetry transformation because its variation produces surface terms, usually neglected over the homogeneous vacuum but non-vanishing in a topological non-trivial background. This phenomenon simultaneously appears with central charge emergence while the saturation of the Bogomolny bound may be studied in terms of the $N=\frac{1}{2}$ SUSY case, a reduced version of the $N=1$ initial situation [6]. In any case the mass correction analysis in supersymmetric models is normally calculated taking the kink solution of the $(\lambda\phi^4)_{1+1}$ theory [3]. Moreover using different techniques a zero-mass correction to the soliton of the sine-Gordon system has been pointed out [7]. In this paper we claim the non-vanishing first quantum contribution to the soliton rest mass and the explicit calculation is performed according to the technique described in [5].

We then start from a system whose Lagrangian density is the following [3]:

$$L=\frac{1}{2}\left\{(\partial_\mu\phi)^2-\frac{2m^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}\phi}{m}\right)\right]+\mathrm{i}\bar{\Psi}\gamma^\mu\partial_\mu\Psi-m\cos\left(\frac{\sqrt{\lambda}\phi}{2m}\right)\bar{\Psi}\Psi\right\} \tag{1}$$

where ϕ represents a real scalar field while Ψ corresponds to a Majorana fermionic one. Taking the bosonic part we point the trivial vacuum as well as the solution with a topological flavour, in this case the well known sine-Gordon soliton [4]

$$\phi_0=0 \qquad \phi_s(x)=\frac{4m}{\sqrt{\lambda}}\tan^{-1}(\mathrm{e}^{mx}). \tag{2}$$

(In the following we set the soliton velocity v equal to zero, which can be always satisfied by a suitable Lorentz transformation.) Before the direct global mass correction calculation we prefer an independent study for both bosonic and fermionic contributions [5].

Firstly, we write the bosonic normal modes equations built around the classical solutions of (2) to sum up

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+m^2\right)\varphi_0(x)=\omega_{\mathrm{B}0}^2\varphi_0(x) \tag{3a}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+m^2-\frac{2m^2}{\cosh^2 mx}\right)\varphi(x)=\omega_{\mathrm{Bs}}^2\varphi(x). \tag{3b}$$

Now the Schrödinger equation (3*b*) includes an only eigenvalue within the discrete spectrum, namely $\omega_{\mathrm{Bs}}^2=0$, while the continuous part starts from the bosonic rest mass m [8]. As only the energy differences are significant we can express the first quantum bosonic correction to the soliton mass in the following way [4]:

$$\Delta M_{\mathrm{B}}=\frac{1}{2}\sum_{r=0}^{\infty}(\omega_{\mathrm{Bs}r}-\omega_{\mathrm{B}0r}). \tag{4}$$

If we recall that the only discrete eigenvalue $\omega_{\mathrm{Bs}}^2=0$ of (3*b*), the more rigorous version of the ΔM_{B} Casimir energy will be

$$\Delta M_{\mathrm{B}}=\frac{1}{2}\int_0^\infty\frac{\mathrm{d}k}{(2\pi)}\sqrt{k^2+m^2}\frac{\mathrm{d}n_{\mathrm{Bs}}}{\mathrm{d}k}-\frac{1}{2}\int_0^\infty\frac{\mathrm{d}k}{(2\pi)}\sqrt{k^2+m^2}\frac{\mathrm{d}_{\mathrm{B}0}}{\mathrm{d}k} \tag{5}$$

where n_{Bs} and $n_{\mathrm{B}0}$ correspond to the density fluctuation modes of the Schrödinger equations expressed in (3). Indeed very elegant closed formulae were found for this bosonic Casimir energy [9]. To sum up: if the denominated $U(x)$ potential, in this case

$$U(x)=-\frac{2m^2}{\cosh^2 mx} \tag{6}$$

obeys the two following conditions: a 'reflectionless' character for the proper $U(x)$ and the integrability of $(1+|x|)\,U(x)$, perfectly satisfied in the sine-Gordon system, then the explicitly finite expression for ΔM_{B} reduces to [9]

$$\Delta M_{\mathrm{B}}=-\frac{m}{\pi}\sum_r(\sin\theta_r-\theta_r\cos\theta_r) \tag{7}$$

where the sum is extended over the discrete spectrum of the equation (3*b*), namely

$$\theta_r = \cos^{-1}\left(\frac{\omega_{Bs}}{m}\right) \qquad 0 \leqslant \omega_{Bs} \leqslant m. \tag{8}$$

Making good use of (8) with $\theta_0 = \pi/2$ we finally obtain [9]

$$\Delta M_B = -m/\pi. \tag{9}$$

Before the global supersymmetric analysis we pass to a purely fermionic study. The general Majorana equations over the backgrounds marked in (2) respectively correspond to

$$\left(i\sigma_2 \frac{d}{dx} + \beta m\right)\Psi(x) = \omega_{F0}\Psi(x) \tag{10a}$$

$$\left(i\sigma_2 \frac{d}{dx} - \beta m \tanh mx\right)\Psi(x) = \omega_{Fs}\Psi(x) \tag{10b}$$

which we can recast in the general form

$$\left(i\sigma_2 \frac{d}{dx} + \beta W(x)\right)\Psi(x) = \omega_F\Psi(x). \tag{11}$$

Writing the spinor in its two-component form

$$\Psi(x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} \tag{12}$$

equation (11) is reduced to the coupled pair

$$\left(\frac{d}{dx} + W(x)\right)v(x) = \omega_F u(x) \tag{13a}$$

$$\left(-\frac{d}{dx} + W(x)\right)u(x) = \omega_F v(x). \tag{13b}$$

In fact the problem contains a hidden SUSY quantum mechanics character where a simple identification leads to [6]

$$S = \begin{bmatrix} 0 & Q^+ \\ Q & 0 \end{bmatrix} \tag{14a}$$

$$Q^+ v = \omega_F u \qquad Qu = \omega_F v. \tag{14b}$$

Then the Majorana equation (11) adopts the form

$$\begin{bmatrix} 0 & Q^+ \\ Q & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \omega_F \begin{bmatrix} u \\ v \end{bmatrix}. \tag{15}$$

Moreover the two Schrödinger equations obtained through Q^+Q and QQ^+ correspond to

$$\left(-\frac{d^2}{dx^2} + W^2 + \frac{dW}{dx}\right)u(x) = \omega_F^2 u(x) \tag{16a}$$

$$\left(-\frac{d^2}{dx^2} + W^2 - \frac{dW}{dx}\right)v(x) = \omega_F^2 v(x). \tag{16b}$$

Anyway the fermionic correction to the soliton mass should be [4]

$$\Delta M_{\rm F}=-\frac{1}{2}\sum_{r=0}^{\infty}(\omega_{\rm Fsr}-\omega_{\rm F0r}). \tag{17}$$

If we bear in mind that equation (10*b*) exhibits only the $\omega_{\rm Fs}=0$ eigenvalue within the discrete spectrum, a more rigorous version of the Casimir energy (17) is

$$\Delta M_{\rm F}=-\frac{1}{2}\int_0^{\infty}\frac{{\rm d}k}{(2\pi)}\sqrt{k^2+m^2}\,\frac{{\rm d}n_{\rm Fs}}{{\rm d}k}+\frac{1}{2}\int_0^{\infty}\frac{{\rm d}k}{(2\pi)}\sqrt{k^2+m^2}\,\frac{{\rm d}n_{\rm F0}}{{\rm d}k} \tag{18}$$

where $n_{\rm Fs}$, $n_{\rm F0}$ represent the density fluctuation modes of the Majorana equations over the topological solution and the homogeneous vacuum respectively. Returning now to the SUSY quantum mechanics character of the fermionic part, we can find an elegant relation between the density fluctuation modes $n_{\rm F}$ of the Majorana equation and the n_u, n_v densities fluctuation modes associated with the Schrödinger equations (16).

Calling $n_{\rm F}$ the density fluctuation modes associated with the S operator, see the (14*a*) equation, it represents half the density of

$$S^2=\begin{bmatrix} Q^+Q & 0 \\ 0 & QQ^+ \end{bmatrix}. \tag{19}$$

To prove this fact we consider the operator [6]

$$P=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{20}$$

maintaining an anticommutation property with S while it commutes with S^2. For each eigenvalue $\omega_{\rm F}^2$ of S^2 we can find two eigenstates $|\Omega\rangle$ and $P|\Omega\rangle$, only one of which is a positive-frequency eigenstate of S with eigenvalue $\omega_{\rm F}$. If we denote by n_u and n_v the densities associated with the two Schrödinger operators of (16), n_u+n_v being the density of S^2, the final result is

$$n_{\rm F}=\tfrac{1}{2}(n_u+n_v). \tag{21}$$

Moreover for the homogeneous vacuum ϕ_0

$$n_{0u}=n_{0v}=n_{\rm B0} \qquad \text{and} \qquad n_{\rm F0}=n_{\rm B0}=n_0 \tag{22}$$

while over the $\phi_{\rm s}(x)$ we must independently consider the n_{su}, n_{sv} densities. We are about to apply these ideas in order to determine the first quantum contribution to the mass of the SUSY sine-Gordon soliton. The quantum contribution, including both bosonic and fermionic terms, is

$$\Delta M=\frac{1}{2}\sum_{r=0}^{\infty}(\omega_{\rm Bsr}-\omega_{\rm Fsr}). \tag{23}$$

If we recall that in this case the normal modes equation (3*b*) coincides with (16*a*), the relations (21) and (22) lead to

$$\Delta M=\frac{1}{4}\int_0^{\infty}\frac{{\rm d}k}{(2\pi)}\sqrt{k^2+m^2}\left(\frac{{\rm d}n_{su}}{{\rm d}k}-\frac{{\rm d}n_0}{{\rm d}k}\right)-\frac{1}{4}\int_0^{\infty}\frac{{\rm d}k}{(2\pi)}\sqrt{k^2+m^2}\left(\frac{{\rm d}n_{sv}}{{\rm d}k}-\frac{{\rm d}n_0}{{\rm d}k}\right) \tag{24}$$

or, in a more simple form,

$$\Delta M=\Delta M_u+\Delta M_v. \tag{25}$$

Applying now a general formula, like the one pointed out in (7) for both ΔM_u, ΔM_v typical bosonic contributions, we get

$$\Delta M_u = -m/2\pi \qquad \Delta M_v = 0 \tag{26}$$

and finally

$$\Delta M = -m/2\pi. \tag{27}$$

Then we have established the non-zero quantum contribution to the soliton mass in the SUSY sine-Gordon model using an explicitly finite method which only needs the discrete levels of Schrödinger equations.

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